ON MODULES WITH REDUCIBLE COMPLEXITY

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ABSTRACT. In this paper we generalize a result, concerning a depth equality over local rings, proved independently by Araya and Yoshino, and Iyengar. Our result exploits complexity, a concept which was initially defined by Alperin for finitely generated modules over group algebras, introduced and studied in local algebra by Avramov, and subsequently further developed by Bergh.

1. INTRODUCTION

Throughout R denotes a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k, and $\operatorname{mod} R$ denotes the category of all finitely generated R-modules.

In this paper we are mainly concerned with the following theorem of Auslander [4]:

Theorem 1.1. ([4, 3.1]) *Let $M,N \in \operatorname{mod} R$ be modules, either of which has finite projective dimension. If $\operatorname{Tor}_i^R(M,N)=0$ for all $i \geq 1$, then it follows that* $\operatorname{depth}(M)+\operatorname{depth}(N)=\operatorname{depth}(R)+\operatorname{depth}(M\otimes_R N)$.

Huneke and Wiegand extended Auslander's result, and proved in [14] that Tor-independent modules (not necessarily of finite projective dimension) over complete intersection rings also satisfy the depth equality of Theorem 1.1; such depth equality was dubbed "the depth formula" by Huneke and Wiegand in [14].

The aforementioned result of Huneke and Wiegand was extended – independently by Araya and Yoshino [3], and Iyengar [15] – to the case where the ring in question is local and either of the modules considered has finite complete intersection dimension; see also Christensen and Jorgensen [11], Foxby [12] and Iyengar [15] for extensions of the depth formula to certain complexes of modules.

The main purpose of this article is to prove an extension of Theorem 1.1. Our main result is:

Theorem 1.2. *Let $M,N \in \operatorname{mod} R$ be modules. Assume $\operatorname{Ext}_R^i(M,R)=0$ for $i \gg 0$ and M has reducible complexity. If $\operatorname{Tor}_i^R(M,N)=0$ for all $i \geq 1$, then* $\operatorname{depth}(M)+\operatorname{depth}(N)=\operatorname{depth}(R)+\operatorname{depth}(M\otimes_R N)$*, i.e., the depth formula for M and N holds.*

In the next section, we recall the definition of complexity and that of reducible complexity, and prove Theorem 1.2 in section 3. Here let us note that the extension of Theorem 1.1 we establish in Theorem 1.2 seems to be quite different in nature than those exist in the literature: all of the improvements of Theorem 1.1, which we are aware of, assume the finiteness of a version of a homological dimension of the module in question. On the contrary, in Theorem 1.2, what we assume for the module M is not a homological dimension. Moreover, our hypothesis on M is weaker than the condition "M has finite complete intersection dimension ". In general, if M has finite complete intersection dimension (e.g., R is a complete intersection), then $\operatorname{Ext}_R^i(M,R)=0$ for $i \gg 0$ and M has reducible complexity, but not vice versa: there do exist examples of

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modules M over Gorenstein rings (so that $\operatorname{Ext}^i_R(M,R)=0$ for $i \gg 0$) such that M has reducible complexity, but M does not have finite complete intersection dimension; see, for example, [9, Example on page 136].

2. Preliminaries

We refer the reader to [5, 8] for the definitions of standard homological dimensions, such as the complete intersection dimension, and proceed by recalling the definitions of Auslander transpose and complexity.

2.1. Auslander Transpose. ([5, 2.8]) Let M be an R-module. Then the *transpose* $\mathsf{Tr}M$ of M is given by the exact sequence $0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \mathsf{Tr}M \to 0$, where $(-)^* = \operatorname{Hom}_R(-,R)$ and $P_1 \xrightarrow{f} P_0 \to M \to 0$ is a projective presentation of M. Notice, $\mathsf{Tr}M$ is unique, up to projectives. Moreover, there is an exact sequence of functors of the form:

(2.1.1) $0 \to \operatorname{Ext}^1_R(\mathsf{Tr}\Omega^n M,-) \to \operatorname{Tor}^R_n(M,-) \to \operatorname{Hom}_R(\operatorname{Ext}^n_R(M,R),-) \to \operatorname{Ext}^2_R(\mathsf{Tr}\Omega^n M,-)$. □

2.2. Complexity. ([1, 2, 6, 7]) If $B=\{b_i\}_{i\geq 0}$ is a sequence of nonnegative integers, then the *complexity* of the sequence B is $\operatorname{cx}(B)=\inf\{r \in \mathbb{N}\cup\{0\} \mid b_n \leq A\cdot n^{r-1}$ for some real number A and for all $n \gg 0\}$.

The complexity $\operatorname{cx}(M,N)$ of a pair of modules $M,N \in \operatorname{mod} R$ is $\operatorname{cx}\left(\{\operatorname{rank}_k(\operatorname{Ext}^i_R(M,N)\otimes_R k)\}\right)$. Then the complexity $\operatorname{cx}(M)$ of M equals $\operatorname{cx}(M,k)$ so that it is a measure on a polynomial scale of the growth of the ranks of the free modules in its minimal free resolution; see [7]. If $M \in \operatorname{mod} R$ has finite complete intersection dimension (e.g., R is a complete intersection), then $\operatorname{cx}_R(M) \leq \operatorname{embdim}(R) - \operatorname{depth}(R)$. □

2.3. Weak Reducible Complexity. ([9]) Let $M,N \in \operatorname{mod} R$. Consider a homogeneous element η of positive degree in the graded module $\operatorname{Ext}^*_R(M,N)=\bigoplus_{i=0}^{\infty}\operatorname{Ext}^i_R(M,N)$. Then choose a map $f_\eta : \Omega^{|\eta|}(M) \to N$ representing η, where $\Omega(M)$ denotes the syzygy of M and $|\eta|$ denotes the degree of η in $\operatorname{Ext}^*_R(M,N)$. This yields a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{|\eta|}(M) & \longrightarrow & F_{|\eta|-1} & \longrightarrow & \Omega^{|\eta|-1}(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_\eta} & & \downarrow & & \downarrow{\scriptstyle \parallel} & & \\
0 & \longrightarrow & N & \longrightarrow & K_\eta & \longrightarrow & \Omega^{|\eta|-1}(M) & \longrightarrow & 0.
\end{array}$$

Here K_η is the pushout of f_η and the inclusion $\Omega^{|\eta|}(M) \hookrightarrow F_{|\eta|-1}$. Note the module K_η is independent, up to isomorphism, of the map f_η chosen to represent η.

The full subcategory of $\operatorname{mod} R$ consisting of modules having *weak-reducible complexity* is defined inductively as follows:

(i) Each module in $\operatorname{mod} R$ of finite projective dimension has weak-reducible complexity.

(ii) If $X \in \operatorname{mod} R$ is a module with $0 < \operatorname{cx}_R(X) < \infty$, then X has weak-reducible complexity provided that there exists a homogeneous element $\eta \in \operatorname{Ext}^*_R(X,X)$, of positive degree, such that $\operatorname{cx}_R(K_\eta) < \operatorname{cx}_R(X)$, and K_η has weak-reducible complexity. □

2.4. Reducible Complexity. ([9]) A module $X \in \operatorname{mod} R$ has *reducible complexity* if it has weak-reducible complexity and $\operatorname{depth}_R(M) = \operatorname{depth}_R(K_\eta)$, where K_η is the module discussed in 2.3. Therefore, over Cohen-Macaulay local rings, the class of modules having weak reducible complexity coincide with the class of modules with reducible complexity. □

2.5. Complete Intersection Dimension Versus Reducible Complexity. If $M \in \operatorname{mod} R$ has finite complete intersection dimension, then it has reducible complexity; see [9, 2.2(i)]. On the other hand, there are modules $M \in \operatorname{mod} R$ having reducible complexity with infinite complete intersection dimension satisfying $\operatorname{Ext}^i_R(M,R) = 0$ for all $i \gg 0$; see for example [9, Example on page 136] and [13, 3.1]. $\square$

3. MAIN RESULT

Bergh [9, 2.2(ii)] showed that, if R is a Cohen-Macaulay local ring and $M \in \operatorname{mod} R$ has reducible complexity, then so does $\Omega^i(M)$ for each $i \geq 0$; his argument in fact implies that the Cohen-Macaulay assumption can be removed for certain values of i. More precisely, Bergh's result implies:

3.1. ([9, 2.2(ii)]) Let $M \in \operatorname{mod} R$ be a module that has weak-reducible complexity.

(i) Then $\Omega^i(M)$ has weak-reducible complexity for each integer $i \geq 0$.
(ii) If $t = \operatorname{depth}(R) - \operatorname{depth}_R(M) \geq 2$, then $\Omega^i(M)$ has reducible complexity for each $i = 1, \dots, t-1$. $\square$

We will also need another result of Bergh:

3.2. ([9, 3.1]) Let $M \in \operatorname{mod} R$ be a module that has reducible complexity. If $\operatorname{Ext}^i_R(M,R) = 0$ for all $i \gg 0$, then it follows that $\operatorname{depth}(R) - \operatorname{depth}_R(M) = \sup\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_R(M,R) \neq 0\}$. $\square$

The proof of Theorem 3.4 relies on the following technical result whose proof is quite involved, and hence deferred to the end of this section.

Lemma 3.3. Let $M, N \in \operatorname{mod} R$ be nonzero modules. Assume M has weak-reducible complexity. Assume further $\operatorname{Tor}^R_i(M,N) = 0 = \operatorname{Ext}^i_R(M,R)$ for all $i \geq 1$. Then it follows $\operatorname{depth}_R(M \otimes_R N) = \operatorname{depth}_R(N)$ and that $\operatorname{Ext}^i_R(\operatorname{Tr} M, N) = 0$ for all $i \geq 1$. $\square$

Next is our main result, which is a generalization of Theorem 1.1 advertised in the introduction. Recall that, if R is a local ring and $M \in \operatorname{mod} R$ is a module with $\operatorname{CI-dim}_R(M) < \infty$, then M has reducible complexity and $\operatorname{Ext}^i_R(M,R) = 0$ for all $i \gg 0$, but not vice versa, in general.

Theorem 3.4. *Let $M, N \in \operatorname{mod} R$ be modules. Assume $\operatorname{Ext}^j_R(M,R) = 0$ for all $j \gg 0$. Assume further M has reducible complexity. If $\operatorname{Tor}^R_i(M,N) = 0$ for all $i \geq 1$, then the depth formula for M and N holds, i.e.,*

$$\operatorname{depth}_R(M) + \operatorname{depth}_R(N) = \operatorname{depth}(R) + \operatorname{depth}_R(M \otimes_R N).$$

Proof. We may assume both M and N are nonzero. Set $t = \operatorname{depth} R - \operatorname{depth}_R(M)$, and proceed by induction on t. Note that, by 3.2, we have $t = \sup\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_R(M,R) \neq 0\}$. Moreover, we may assume $t \geq 1$ as if $t = 0$, then the assertion follows from Lemma 3.3.

Now we argue by induction on $\operatorname{cx}_R(M)$. If $\operatorname{cx}_R(M) = 0$, then $\operatorname{pd}_R(M) < \infty$, and so the depth formula holds by Theorem 1.1. Hence we assume assume $\operatorname{pd}_R(M) = \infty$, i.e., $\operatorname{cx}_R(M) \geq 1$. As M has reducible complexity, there exists a short exact sequence

$$0 \to M \to K \to \Omega^n(M) \to 0, \tag{3.4.1}$$

where n is a nonnegative integer, $K \in \operatorname{mod} R$ has reducible complexity, $\operatorname{cx}_R(K) < \operatorname{cx}_R(M)$ and $\operatorname{depth}_R(K) = \operatorname{depth}_R(M)$. Note, it follows from (3.4.1) that $\operatorname{Ext}^j_R(K,R) = 0$ for all $j \gg 0$, and $\operatorname{Tor}^R_i(K,N) = 0$ for all $i \geq 1$. So the induction hypothesis on the complexity gives the equality:

$$\operatorname{depth}_R(K) + \operatorname{depth}_R(N) = \operatorname{depth}(R) + \operatorname{depth}_R(K \otimes_R N). \tag{3.4.2}$$

Note, since $\operatorname{Tor}^R_i(M,N) = 0$ for all $i \geq 1$, tensoring (3.4.1) with N, we obtain the exact sequence:

$$0 \to M \otimes_R N \to K \otimes_R N \to \Omega^n(M) \otimes_R N \to 0. \tag{3.4.3}$$

Next we will consider cases for the nonnegative integer n:

Case 1. Assume $n = 0$. Then $\Omega^n(M) = M$, and the depth lemma applied to the short exact sequence (3.4.3) yields $\operatorname{depth}_R(M \otimes_R N) = \operatorname{depth}_R(K \otimes_R N)$. So, the depth formula for M and N holds by (3.4.2).

For the remaining cases, we will make use of the following observation; it follows easily from the depth lemma and (3.4.3).

(3.4.4) The proof of the theorem is complete in case $\operatorname{depth}_R(\Omega^n(M) \otimes_R N) > \operatorname{depth}_R(K \otimes_R N)$.

Case 2. Assume $n \geq t$. Notice, by 3.1, $\Omega^n(M)$ has weak-reducible complexity. Hence Lemma 3.3 implies that:

$$\operatorname{depth}_R(\Omega^n(M) \otimes_R N) = \operatorname{depth}_R(N). \tag{3.4.8}$$

Therefore, since $t \geq 1$, (3.4.2) and (3.4.8) yield that:

$$\operatorname{depth}_R(K \otimes_R N) < \operatorname{depth}_R(K \otimes_R N) + t = \operatorname{depth}_R(N) = \operatorname{depth}_R(\Omega^n(M) \otimes_R N).$$

Hence the required result follows due to (3.4.4).

Case 3. Assume $1 \leq n \leq t-1$. In this case, by 3.1, we know $\Omega^n(M)$ has reducible complexity. Since $\operatorname{depth}(R) - \operatorname{depth}_R(M) = t \geq 2$, we have $\operatorname{depth}_R(\Omega^n(M)) = \operatorname{depth}_R(M) + v$ for some positive integer v with $1 \leq v \leq t-1$. Hence, $\operatorname{depth}(R) - \operatorname{depth}_R(\Omega^n(M)) = t - v < t$. Now, by replacing the pair (M,N) with $(\Omega^n(M), N)$, and by using the induction hypothesis on t, we obtain:

$$\operatorname{depth}_R(\Omega^n(M)) + \operatorname{depth}_R(N) = \operatorname{depth}(R) + \operatorname{depth}_R(\Omega^n(M) \otimes_R N). \tag{3.4.5}$$

Thus, since $\operatorname{depth}_R(\Omega^n(M)) = \operatorname{depth}_R(M) + v$, we conclude from (3.4.2) and (3.4.5) that:

$$\operatorname{depth}_R(\Omega^n(M) \otimes_R N) = v + \operatorname{depth}(N) - t = v + \operatorname{depth}(K \otimes_R N). \tag{3.4.6}$$

In particular, we see from (3.4.6) that:

$$\operatorname{depth}_R(\Omega^n(M) \otimes_R N) > \operatorname{depth}_R(K \otimes_R N). \tag{3.4.7}$$

Thus the proof of Case 3, as well as the proof of the theorem, is complete by (3.4.4).

□

We now proceed to establish Lemma 3.3 and complete the proof of Theorem 3.4. For that we will make use of the following results, which are recorded here for the convenience of the reader.

3.5. ([3, 4.1]) Let $X, Y \in \operatorname{mod} R$ be modules such that $\operatorname{Ext}^i_R(X,Y) = 0$ for all $i \geq 1$. Then it follows that $\operatorname{depth}_R(\operatorname{Hom}_R(X,Y)) = \operatorname{depth}_R(Y)$. □

3.6. ([5, 3.9]) Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\operatorname{mod}(R)$. Then it follows that the sequence $0 \to C^* \to B^* \to C^* \to \operatorname{Tr} A \to \operatorname{Tr} B \to \operatorname{Tr} C \to 0$ is exact. □

3.7. Let $M, N \in \operatorname{mod} R$ be modules and let $n \geq 1$ be an integer. Assume $\operatorname{Ext}^i_R(M,R) = 0$ for all $i \geq n$. Then, for each integer j with $j \geq n$, we have $\operatorname{Ext}^j_R(M,N) \cong \operatorname{Tor}^R_1(\operatorname{Tr}\Omega^j M, N)$ and $\operatorname{Tr}\Omega^{j-1} M \cong \Omega \operatorname{Tr} \Omega^j M$ (up to free summands); see [5, 2.8] for details. □

3.8. ([9, 2.3 and 2.4(i)]; see also [10, 2.1(ii)]) Let $M \in \operatorname{mod} R$ and let $\eta \in \operatorname{Ext}^{|\eta|}_R(M,M)$ be an element.

(i) There is an exact sequence $0 \to \Omega^{|\eta|}(K_\eta) \to K_{\eta^2} \oplus F \to K_\eta \to 0$ in $\operatorname{mod} R$, where F is a free module.

(ii) Assume K_η reduces the complexity of M. Then it follows that:

$$\mathrm{cx}_R(K_{\eta^2}) = \mathrm{cx}_R(K_{\eta^2} \oplus F) \leq \max\{\mathrm{cx}_R(\Omega^{|\eta|}(K_\eta)), \mathrm{cx}_R(K_\eta)\} = \mathrm{cx}_R(K_\eta) < \mathrm{cx}_R(M).$$

Therefore, there is an exact sequence of the form $0 \to M \to K_{\eta^2} \to \Omega^{2|\eta|-1}(M) \to 0$, where K_{η^2} also reduces the complexity of M. □

Remark 3.9. In [9, 2.4(i)] it is assumed that the ring in question is a complete intersection. Also, in [10, 2.1(ii)], it is assumed that the module M considered has finite complete intersection dimension. Although we refer to [9, 2.4(i)] (or [10, 2.1(ii)]) in the proof of Lemma 3.3, we do not need that rings are complete intersections or modules have finite complete intersection dimension in the context of our argument; see 3.8. □

A Proof of Lemma 3.3. We set $c = \mathrm{cx}_R(M)$, and proceed by induction on c.

Assume $c = 0$, i.e., $\mathsf{pd}_R(M) < \infty$. Then, since $\mathrm{Ext}^i_R(M,R) = 0$ for all $i \geq 1$, it follows that M is free. Therefore, $\mathsf{Tr}M = 0$ and the claim follows.

Next assume $c \geq 1$. As $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$, it follows from (2.1.1) that

(3.3.1) $$\mathrm{Ext}^1_R(\mathsf{Tr}\Omega^i M, N) = 0 \text{ for all } i \geq 1.$$

Moreover, since $\mathrm{Ext}^i_R(M,R) = 0$ for all $i \geq 1$, the following stable isomorphism is deduced from 3.7:

(3.3.2) $$\mathsf{Tr}\Omega^{u-v}M \cong \Omega^v\mathsf{Tr}\Omega^u M, \text{ for all positive integers } u \text{ and } v \text{ with } u \geq v.$$

Therefore, for a given integer $t \geq 2$ and $1 \leq j \leq t-1$, we have that:

(3.3.3) $$\mathrm{Ext}^j_R(\mathsf{Tr}\Omega^{t-1}M, N) \cong \mathrm{Ext}^1_R(\Omega^{j-1}\mathsf{Tr}\Omega^{t-1}M, N) \cong \mathrm{Ext}^1_R(\mathsf{Tr}\Omega^{t-j}M, N) = 0.$$

The second isomorphism and the first equality in (3.3.3) are due to (3.3.2) and (3.3.1), respectively.

Now let $\eta \in \mathrm{Ext}^*_R(M,M)$ be an element reducing the complexity of M; see 2.3. Hence, there is an exact sequence of the form:

(3.3.4) $$0 \to M \to K \to \Omega^q(M) \to 0,$$

where $q = |\eta| - 1$, $K = K_\eta$, $\mathrm{cx}_R(K) < c$ and K has weak-reducible complexity. As $\mathrm{Tor}_i^R(M,N) = 0 = \mathrm{Ext}^i_R(M,R)$ for all $i \geq 1$, it follows from (3.3.4) that $\mathrm{Ext}^i_R(K,R) = 0 = \mathrm{Tor}_i^R(K,N)$ for all $i \geq 1$. So, by the induction hypothesis, we conclude:

(3.3.5) $$\mathrm{Ext}^i_R(\mathsf{Tr}K, N) = 0 \text{ for all } i \geq 1, \text{ and } \mathrm{depth}_R(K \otimes_R N) = \mathrm{depth}_R(N).$$

We proceed to prove the required assertions, i.e., the vanishing of $\mathrm{Ext}^i_R(\mathsf{Tr}M,N)$ for all $i \geq 1$ and the depth equality $\mathrm{depth}_R(M \otimes_R N) = \mathrm{depth}_R(N)$, in several steps.

Claim 1. We have that $\mathrm{Ext}^i_R(\mathsf{Tr}\Omega^q(M),N) \cong \mathrm{Ext}^{i+1}_R(\mathsf{Tr}M,N) \cong \mathrm{Ext}^{i+q+1}_R(\mathsf{Tr}\Omega^q(M),N)$ for all $i \geq 1$.

Proof of Claim 1. The short exact sequence (3.3.4), in view of 3.6, yields the exact sequence:

(3.3.6) $$0 \to (\Omega^q(M))^* \to K^* \to M^* \to \mathsf{Tr}\Omega^q(M) \to \mathsf{Tr}K_\eta \to \mathsf{Tr}M \to 0.$$

Since $\mathrm{Ext}^1_R(\Omega^q(M),R) = 0$, the following sequence is exact:

(3.3.7) $$0 \to \mathsf{Tr}\Omega^q(M) \to \mathsf{Tr}K \to \mathsf{Tr}M \to 0.$$

We obtain, by applying $\mathrm{Hom}_R(-,N)$ to (3.3.7), the following long exact sequence:

(3.3.8) $$\cdots \to \mathrm{Ext}^i_R(\mathsf{Tr}M,N) \to \mathrm{Ext}^i_R(\mathsf{Tr}K,N) \to \mathrm{Ext}^i_R(\mathsf{Tr}\Omega^q(M),N) \to \cdots$$

Now (3.3.8) and (3.3.5) give:

$$\operatorname{Ext}_R^{i+1}(\mathsf{Tr}M,N) \cong \operatorname{Ext}_R^{i}(\mathsf{Tr}\Omega^q(M),N) \text{ for all } i \geq 1. \tag{3.3.9}$$

Consequently, for all $i \geq 1$, we establish:

$$\operatorname{Ext}_R^{i}(\mathsf{Tr}\Omega^q(M),N) \cong \operatorname{Ext}_R^{i+1}(\mathsf{Tr}M,N) \cong \operatorname{Ext}_R^{i+1}(\Omega^q\mathsf{Tr}\Omega^q(M),N) \cong \operatorname{Ext}_R^{i+q+1}(\mathsf{Tr}\Omega^q(M),N). \tag{3.3.10}$$

Here, in (3.3.10), the first and second isomorphisms are due to (3.3.9) and (3.3.2), respectively. This completes the proof of Claim 1. □

Claim 2. We have that $\operatorname{Ext}_R^{i}(\mathsf{Tr}\Omega^q(M),N) \cong \operatorname{Ext}_R^{i+j(q+1)}(\mathsf{Tr}\Omega^q(M),N)$ for all $i \geq 1$ and $j \geq 1$.
Proof of Claim 2. This follows by repeated applications of Claim 1. □

Claim 3. We have that $\operatorname{Ext}_R^{i}(\mathsf{Tr}\Omega^{2q+1}(M),N) \cong \operatorname{Ext}_R^{i+1}(\mathsf{Tr}M,N) \cong \operatorname{Ext}_R^{2q+i+2}(\mathsf{Tr}\Omega^{2q+1}M,N)$ for all $i \geq 1$.
Proof of Claim 3. It follows that η^2 reduces the complexity of M, and there are exact sequences:

$$0 \to M \to Z \to \Omega^{2q+1}(M) \to 0, \tag{3.3.11}$$

and

$$0 \to \Omega^{q+1}(K) \to Z \oplus F \to K \to 0, \tag{3.3.12}$$

where $Z = K_{\eta^2}$ and F is a free module; see 2.3 and 3.8.

As $\operatorname{Ext}_R^1(K,R) = 0$, the following exact sequence follows from (3.3.12) and 3.6:

$$0 \to \mathsf{Tr}K \to \mathsf{Tr}Z \to \mathsf{Tr}\Omega^{q+1}(K) \to 0. \tag{3.3.13}$$

Applying $\operatorname{Hom}_R(-,N)$ to the exact sequence (3.3.13), we get a long exact sequence:

$$\cdots \to \operatorname{Ext}_R^{i}(\mathsf{Tr}\Omega^{q+1}(K),N) \to \operatorname{Ext}_R^{i}(\mathsf{Tr}Z,N) \to \operatorname{Ext}_R^{i}(\mathsf{Tr}K,N) \to \cdots \tag{3.3.14}$$

Note that $\Omega^{q+1}(K)$ has weak-reducible complexity; see 3.1(i). Note also $\operatorname{cx}_R(\Omega^{q+1}(K)) = \operatorname{cx}_R(K) < c$, and $\operatorname{Ext}_R^{i}(\Omega^{q+1}(K),R) = 0 = \operatorname{Tor}_i^R(\Omega^{q+1}(K),N)$ for all $i \geq 1$. Therefore, by the induction hypothesis on c, we have that $\operatorname{Ext}_R^{i}(\mathsf{Tr}\Omega^{q+1}(K),N) = 0$ for all $i \geq 1$. In view of (3.3.5) and (3.3.14), we conclude:

$$\operatorname{Ext}_R^{i}(\mathsf{Tr}Z,N) = 0 \text{ for all } i \geq 1. \tag{3.3.15}$$

The short exact sequence (3.3.11) and 3.6 yield the following exact sequence:

$$0 \to (\Omega^{2q+1}M)^* \to (Z)^* \to M^* \to \mathsf{Tr}\Omega^{2q+1}(M) \to \mathsf{Tr}Z \to \mathsf{Tr}M \to 0. \tag{3.3.16}$$

Since we have $\operatorname{Ext}_R^{2q+2}(M,R) = 0$, by (3.3.16), we get the exact sequence:

$$0 \to \mathsf{Tr}\Omega^{2q+1}(M) \to \mathsf{Tr}Z \to \mathsf{Tr}M \to 0. \tag{3.3.17}$$

Now (3.3.17) induces the long exact sequence for all $i \geq 1$:

$$\cdots \to \operatorname{Ext}_R^{i}(\mathsf{Tr}M,N) \to \operatorname{Ext}_R^{i}(\mathsf{Tr}Z,N) \to \operatorname{Ext}_R^{i}(\mathsf{Tr}\Omega^{2q+1}(M),N) \to \cdots \tag{3.3.18}$$

Consequently, for all $i \geq 1$, we have:

$$\begin{aligned}\operatorname{Ext}_R^{i}(\mathsf{Tr}\Omega^{2q+1}(M),N) &\cong \operatorname{Ext}_R^{i+1}(\mathsf{Tr}M,N)\\ &\cong \operatorname{Ext}_R^{i+1}(\Omega^{2q+1}\mathsf{Tr}\Omega^{2q+1}(M),N)\\ &\cong \operatorname{Ext}_R^{2q+i+2}(\mathsf{Tr}\Omega^{2q+1}M,N)\end{aligned} \tag{3.3.19}$$

Here, in (3.3.19), the first isomorphism follows from the long exact sequence in (3.3.18) since $\mathrm{Ext}^i_R(\mathsf{Tr}Z,N)$ vanishes for all $i\geq 1$; see (3.3.15). Furthermore, the second isomorphism of (3.3.19) is due to (3.3.2). This completes the proof of Claim 3. $\square$

Claim 4. Assume $q\geq 1$. Then, given $j\geq 1$, we have that $\mathrm{Ext}^i_R(\mathsf{Tr}\Omega^q(M),N)=0$ for all $i\neq j(q+1)$, i.e., $\mathrm{Ext}^i_R(\mathsf{Tr}\Omega^q(M),N)=0$ for all i, where $(j-1)q+j\leq i\leq jq+(j-1)$.

Proof of Claim 4. Let $j\geq 1$ be an integer.

If $j=1$, then setting $t=q+1$ in (3.3.3), we see that $\mathrm{Ext}^i_R(\mathsf{Tr}\Omega^q(M),N)=0$ for all i with $1\leq i\leq q$. Hence assume $j\geq 2$. In this case, we have $1\leq i-(j-1)(q+1)\leq q$, and Claim 2 implies that:

$$\mathrm{Ext}^i_R(\mathsf{Tr}\Omega^q(M),N)\cong \mathrm{Ext}^{i-(j-1)(q+1)}_R(\mathsf{Tr}\Omega^q(M),N) \tag{3.3.20}$$

We have observed $\mathrm{Ext}^v_R(\mathsf{Tr}\Omega^q(M),N)=0$ for all v with $1\leq v\leq q$. Thus, since $1\leq i-(j-1)(q+1)\leq q$, we see that $\mathrm{Ext}^{i-(j-1)(q+1)}_R(\mathsf{Tr}\Omega^q(M),N)=0$. Therefore Claim 4 follows from (3.3.20). $\square$

Claim 5. If $q\geq 1$, then we have that $\mathrm{Ext}^i_R(\mathsf{Tr}\Omega^q M,N)=0$ for all $i\geq 1$.

Proof of Claim 5. We have:

$$0=\mathrm{Ext}^{q+1}_R(\mathsf{Tr}\Omega^{2q+1}(M),N)\cong \mathrm{Ext}^{q+2}_R(\mathsf{Tr}M,N)\cong \mathrm{Ext}^{q+1}_R(\mathsf{Tr}\Omega^q(M),N) \tag{3.3.21}$$

Here, in (3.3.21), the first equality follows from (3.3.3) by letting $t=2q+2$ and $j=q+1$. Furthermore, the first and second isomorphisms are due to Claim 1 and Claim 3 (with $i=q+1$), respectively.

Claim 2, in view of (3.3.21), implies that $0=\mathrm{Ext}^{q+1}_R(\mathsf{Tr}\Omega^q(M),N)\cong \mathrm{Ext}^{(q+1)+j(q+1)}_R(\mathsf{Tr}\Omega^q(M),N)$ for all $j\geq 1$, i.e., $\mathrm{Ext}^{r(q+1)}_R(\mathsf{Tr}\Omega^q(M),N)=0$ for all $r\geq 1$. This observation, in combination with Claim 4, establishes Claim 5. $\square$

Claim 6. We have that $\mathrm{Ext}^i_R(\mathsf{Tr}M,N)=0$ for all $i\geq 1$.

Proof of Claim 6. Assume first $q=0$. Then, for all $i\geq 1$, we have:

$$\mathrm{Ext}^{i+1}_R(\mathsf{Tr}M,N)\cong \mathrm{Ext}^i_R(\mathsf{Tr}M,N)\cong \mathrm{Ext}^i_R(\mathsf{Tr}\Omega M,N) \tag{3.3.22}$$

Here, in (3.3.22), the first and the second isomorphism follows from Claim 1 and Claim 3, respectively. Since $\mathrm{Ext}^1_R(\mathsf{Tr}\Omega M,N)$ vanishes due to (3.3.1), we conclude that $\mathrm{Ext}^i_R(\mathsf{Tr}M,N)=0$ for all ≥ 1.

Next assume $q\geq 1$. Then, for all $i\geq 1$, we have:

$$0=\mathrm{Ext}^i_R(\mathsf{Tr}\Omega^q(M),N)\cong \mathrm{Ext}^{i+1}_R(\mathsf{Tr}M,N) \tag{3.3.22}$$

In (3.3.22), the first equality is due to Claim 5, while the first isomorphism follows from Claim 1. Consequently, we have $\mathrm{Ext}^i_R(\mathsf{Tr}M,N)=0$ for all $i\geq 2$. Furthermore, it follows:

$$\mathrm{Ext}^1_R(\mathsf{Tr}M,N)\cong \mathrm{Ext}^1_R(\Omega^q\mathsf{Tr}\Omega^q M,N)\cong \mathrm{Ext}^{q+1}_R(\mathsf{Tr}\Omega^q M,N)=0 \tag{3.3.23}$$

The first isomorphism of (3.3.23) is due to (3.3.2), and the first equality is from Claim 5. This proves the vanishing of $\mathrm{Ext}^i_R(\mathsf{Tr}M,N)$ for all $i\geq 1$, and completes the proof of Claim 6. $\square$

Claim 7. We have that $\mathrm{depth}_R(M\otimes_R N)=\mathrm{depth}_R(N)$.

Proof of Claim 7. Recall that $M^*\cong \Omega^2\mathsf{Tr}M\oplus G$ for some free module $G\in \mathsf{mod}R$; see 2.1. Therefore, as Claim 6 shows $\mathrm{Ext}^i_R(\mathsf{Tr}M,N)=0$ for all ≥ 1, we conclude that $\mathrm{Ext}^i_R(M^*,N)=0$ for all ≥ 1. This implies, in view of 3.5, that:

$$\mathrm{depth}_R(\mathrm{Hom}(M^*,N))=\mathrm{depth}_R(N) \tag{3.3.24}$$

On the other hand, since $\mathrm{Ext}^1_R(\mathsf{Tr}M,N)=0=\mathrm{Ext}^2_R(\mathsf{Tr}M,N)=0$, setting $n=0$, we obtain from 2.1.1 that:

$$M\otimes_R N\cong \mathrm{Hom}_R(M^*,N) \tag{3.3.25}$$

Consequently, the proof of Claim 7 is complete due to (3.3.24) and (3.3.25). □

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